Bounds on the algebraic degree of iterated constructions

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DTU Compute

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Algebraic degree of a vectorial function $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$

Example (ANF of a permutation $F$ of $\mathbb{F}_2^4$)

$$(y_0, y_1, y_2, y_3) = F(x_0, x_1, x_2, x_3)$$

$$y_0 = x_0x_2 + x_1 + x_2 + x_3$$
$$y_1 = x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2 + x_0x_3 + x_2x_3 + x_0 + x_2$$
$$y_2 = x_0x_1x_3 + x_0x_2x_3 + x_1x_2 + x_1x_3 + x_2x_3 + x_0 + x_1 + x_3$$
$$y_3 = x_0x_1x_2 + x_1x_3 + x_0 + x_1 + x_2 + 1.$$
Iterated permutations

Most of the symmetric constructions (hash functions, block ciphers) are based on a permutation iterated a high number of times.

Important to estimate the algebraic degree of such iterated permutations.

Functions with a low degree are vulnerable to:

- Algebraic attacks
- Higher-order differential attacks and distinguishers
- Cube attacks
Higher-order derivatives

Let $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$.
Derivative of $F$ at $a \in \mathbb{F}_2^n$: $D_a(x) = F(x) \oplus F(x + a)$.

**Definition.** For any $k$-dimensional subspace $V$ of $\mathbb{F}_2^n$, the $k$-th order derivative of $F$ with respect to $V$ is the function defined by

$$D_V F(x) = D_{a_1} \ldots D_{a_k}(x) = \bigoplus_{v \in V} F(x+v), \quad \text{for every} \quad x \in \mathbb{F}_2^n.$$ 

where $(a_1, \ldots, a_k)$ is a basis of $V$.

**Example:** ($k = 2, V = \langle a, b \rangle$)

$$D_V(x) = D_a D_b(x) = D_a(F(x) \oplus F(x + b)) = F(x) \oplus F(x + a) \oplus F(x + b) \oplus F(x + a + b)$$
Higher-order differential cryptanalysis

Introduced by Knudsen in 1994. Based on the following properties:

Let $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$ of degree $d$.

**Proposition.** For every $a \in \mathbb{F}_2^n$ we have

$$D_a F \leq d - 1.$$

**Proposition.** [Lai 94] For every $V \subset \mathbb{F}_2^n$, with $\dim V > d$

$$D_V(x) = 0, \text{ for every } x \in \mathbb{F}_2^n.$$
The $\mathcal{KN}$ cipher [Knudsen – Nyberg 95]

6-round Feistel cipher

- $\mathcal{E} : \mathbb{F}_2^{32} \to \mathbb{F}_2^{33}$ linear
- $\mathcal{T} : \mathbb{F}_2^{33} \to \mathbb{F}_2^{32}$ linear
- $k_i$: 33-bit subkey
- $S : x \mapsto x^3$ over $\mathbb{F}_2^{33}$

Algebraic degree of $S$: 2
Higher-order differential attack on \( \mathcal{KN} \)

\[ y_0(x) = c \]
\[ y_1(x) = x + F_{k_1}(c) := x + c' \]
\[ y_2(x) = F_{k_2}(x + c') + c \]
\[ y_3(x) = F_{k_3}(F_{k_2}(x + c') + c) + x + c' \]
\[ y_4(x) = F_{k_4}(F_{k_3}(F_{k_2}(x + c') + c) + x + c') + F_{k_2}(x + c') + c \]

\[ G = F_{k_4} \circ F_{k_3} \circ F_{k_2}. \]

\[ \text{deg}(G) \leq 2^3 \]
If $V \subset \mathbb{F}_2^{32}$ with $\dim(V) = 9$, then:

$$D_V y_4(x) = 0, \text{ for all } x \in \mathbb{F}_2^{32}. \quad (1)$$

By definition:

$$\bigoplus_{v \in V} y_4(v + w) = 0, \text{ for all } w \in \mathbb{F}_2^{32}. \quad (1)$$

We can see that:

$$x_6(x) = F_{k_6}(y_6(x)) + y_4(x),$$

and by inverting the terms:

$$y_4(x) = x_6(x) + F_{k_6}(y_6(x)). \quad (2)$$
Key recovery

By combining equations (1) and (2), we obtain the attack equation:

$$\bigoplus_{v \in V} F_{k_6}(y_6(v + w)) + \bigoplus_{v \in V} x_6(v + w) = 0.$$ 

The right subkey $k_6$ is the one for which the equation is verified.

Complexity of the attack:

- **Data Complexity:** $2^9$ plaintexts.
- **Time Complexity:** $2^{33+8}$.

**Distinguisher** for 4 and 5 rounds with data complexity $2^5$ and $2^9$ respectively.
SHA-3 [Bertoni – Daemen – Peeters – VanAssche 08]

Sponge construction

**Keccak-\( f \) Permutation**

- 1600-bit state, seen as a 3-dimensional 5 \( \times \) 5 \( \times \) 64 matrix
- 24 rounds \( R \)
- **Nonlinear layer**: 320 parallel applications of a 5 \( \times \) 5 S-box \( \chi \)
- \( \text{deg} \chi = 2, \text{deg} \chi^{-1} = 3 \)
Outline

Some first bounds on the degree

A bound on the degree of SPN constructions

Influence of the inverse permutation
Outline

Some first bounds on the degree

A bound on the degree of SPN constructions

Influence of the inverse permutation
A trivial bound

**Proposition:** Let $F$ be a function from $\mathbb{F}_2^n$ into $\mathbb{F}_2^n$ and $G$ a function from $\mathbb{F}_2^n$ into $\mathbb{F}_2^m$. Then

$$\deg(G \circ F) \leq \deg(G) \deg(F).$$

**Example:** Round function $R$ of AES is of degree 7. Then

$$\deg(R^2) = \deg(R \circ R) \leq 7^2 = 49.$$
A bound based on the Walsh spectrum

[Canteaut – Videau ’02]

**Definition (Walsh spectrum of $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$)**

$$\{ F(\varphi_b \circ F + \varphi_\alpha) = \sum_{x \in \mathbb{F}_2^n} (-1)^{b \cdot F(x) + a \cdot x}, a, b \in \mathbb{F}_2^n, b \neq 0 \}.$$ 

**Theorem:** If all the values in the Walsh spectrum of $F$ are divisible by $2^\ell$, then for every $G : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$

$$\deg(G \circ F) \leq n - \ell + \deg(G).$$
Application to SHA-3

It can be computed that:

- The Walsh spectra of $\chi$ and $\chi^{-1}$ are divisible by $2^3$.

As there are 320 parallel applications of $\chi$ in a round we have:

- The Walsh spectra of $R$ and $R^{-1}$ are divisible by $2^{3 \cdot 320} = 2^{960}$.

Bound for the degree of $R^{-7}$

$$\deg(R^{-7}) = \deg(R^{-6} \circ R^{-1}) \leq 1600 - 960 + \deg(R^{-6}) \leq 1369.$$
Application to SHA-3

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Outline

Some first bounds on the degree

A bound on the degree of SPN constructions

Influence of the inverse permutation
Substitution Permutation Networks

How to estimate the evolution of the degree of such constructions?
After several rounds, all coordinates can be expressed as a sum of monomials.

Each monomial is a **product** of variables in $X = \{x_0, \ldots, x_{15}\}$. 
After several rounds, all coordinates can be expressed as a sum of monomials.

Each monomial is a **product** of variables in \( Y = \{ y_0, \ldots, y_{15} \} \).

The coordinates \( y_0 - y_3 \) are outputs of the same Sbox (equally for the others).

**What is the consequence on the degree of the product?**
The notion of $\delta_k$

**Definition:** For a permutation $S$ define $\delta_k(S)$ as the maximum degree of the product of $k$ coordinates of $S$.

$\rightarrow \delta_1(S) := \text{algebraic degree of } S$

**Example:**

$$\text{deg } S = 3$$

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$$S$$
The notion of $\delta_k$

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Example:

\[
\begin{array}{c|c}
 k & \delta_k \\
 \hline
 1 & 3 \\
 2 & 3 \\
 3 & 3 \\
 4 & 4 \\
\end{array}
\]

$S$ permutation of $F_2^n$:
$\delta_k(S) = n$ iff $k = n$. 

$\deg S = 3$
Example: Product of 6 coordinates.

\[ \pi = y_0y_1y_3y_8y_9y_{10}. \]

\[ \text{deg}(\pi) \leq \delta_3(S_1) + \delta_3(S_3) = 6. \]
**Example:** Product of 6 coordinates.

\[ \pi = y_0 y_5 y_8 y_{10} y_{13} y_{15}. \]

\[ \deg(\pi) \leq \delta_1(S_1) + \delta_1(S_2) + \delta_2(S_3) + \delta_2(S_4) = 12. \]

The degree of the product is relatively low if many coordinates coming from the same Sbox are involved!!!
Towards the bound

Find the maximal degree of the product $\pi$ of $d$ outputs.

$x_i = \# \text{ Sboxes for which exactly } i \text{ coordinates are involved in } \pi.$
Towards the bound

Find the maximal degree of the product $\pi$ of $d$ outputs.

$x_i = \#$ Sboxes for which exactly $i$ coordinates are involved in $\pi$.

**Example** ($d = 13$)
- $x_4 = 1$, $x_3 = 3$:

  \[
  \deg(\pi) \leq \delta_3 x_3 + \delta_4 x_4 = 3 \cdot 3 + 4 \cdot 1 = 13.
  \]
Towards the bound

Find the maximal degree of the product $\pi$ of $d$ outputs.

$x_i = \# \text{ Sboxes for which exactly } i \text{ coordinates are involved in } \pi.$

Example ($d = 13$)

- $x_4 = 2$, $x_3 = 1$, $x_2 = 1$:

$$\deg(\pi) \leq \delta_2 x_2 + \delta_3 x_3 + \delta_4 x_4 = 3 \cdot 1 + 3 \cdot 1 + 4 \cdot 2 = 14.$$
Towards the bound

Find the maximal degree of the product $\pi$ of $d$ outputs.

$x_i = \# \text{Sboxes for which exactly } i \text{ coordinates are involved in } \pi.$

Example ($d = 13$)

- $x_4 = 3, \ x_1 = 1$

$$\deg(\pi) \leq \delta_1 x_1 + \delta_4 x_4 = 3 \cdot 1 + 4 \cdot 3 = 15.$$
Towards the bound

\[
\begin{align*}
S & \quad S & \quad S & \quad S \\
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\end{align*}
\]

Find the maximal degree of the product \( \pi \) of \( d \) outputs.

\[
x_i = \# \text{ Sboxes for which exactly } i \text{ coordinates are involved in } \pi.
\]

\[
\deg(\pi) \leq \max_{(x_1, x_2, x_3, x_4)} (\delta_1 x_1 + \delta_2 x_2 + \delta_3 x_3 + \delta_4 x_4)
\]

with \( x_1 + 2x_2 + 3x_3 + 4x_4 = d \).
Some first bounds on the degree  A bound on the degree of SPN constructions  Influence of the inverse permutation

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<thead>
<tr>
<th>$d$</th>
<th>$x_4$</th>
<th>$x_3$</th>
<th>$x_2$</th>
<th>$x_1$</th>
<th>$\text{deg}(\pi)$</th>
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$$16 - \text{deg}(\pi) \geq \frac{16 - d}{3}$$
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<tr>
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\[
\text{deg}(\pi) \leq 16 - \frac{16 - d}{3}
\]
A bound on the degree of SPN constructions

[ Boura – Canteaut – De Cannière - FSE 2011 ]

**Theorem.** Let $F$ be a function from $\mathbb{F}_2^n$ into $\mathbb{F}_2^n$ corresponding to the parallel application of an Sbox, $S$, defined over $\mathbb{F}_2^{n_0}$. Then, for any $G$ from $\mathbb{F}_2^n$ into $\mathbb{F}_2^{\ell}$, we have

$$\text{deg}(G \circ F) \leq n - \frac{n - \text{deg} G}{\gamma(S)},$$

where

$$\gamma(S) = \max_{1 \leq i \leq n_0 - 1} \frac{n_0 - i}{n_0 - \delta_i}.$$
Application to SHA-3

**Non-linear layer:** Parallel application of a $5 \times 5$ Sbox $\chi$, with $\deg(\chi) = 2$.

$$\gamma(\chi) = \max_{1 \leq k \leq 4} \frac{5 - k}{5 - \delta_k(\chi)}$$

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<tr>
<th>$k$</th>
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<td>2</td>
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<td>4</td>
<td>4</td>
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$$\gamma(\chi) = \max \left( \frac{4}{3}, \frac{3}{1}, \frac{2}{1}, \frac{1}{1} \right) = 3$$

We deduce

$$\deg(G \circ F) \leq 1600 - \frac{1600 - \deg(G)}{3}$$
$R$: Round function of Keccak-$f$

For $r = 11, \ldots, 16$:

\[
\text{deg}(R^r) \leq 1600 - \frac{1600 - \text{deg}(R^{r-1})}{3}
\]

**Example:** $r = 11$

\[
\text{deg}(R^{11}) \leq 1600 - \frac{1600 - \text{deg}(R^{10})}{3} = 1600 - \frac{1600 - 1024}{3} = 1600 - \frac{576}{3} = 1408.
\]

<table>
<thead>
<tr>
<th>$r$</th>
<th>deg($R^r$)</th>
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SPN Bound vs. Trivial Bound

![Graph showing the comparison between SPN Bound and Trivial Bound over rounds.](image-url)
SPN Bound vs. Trivial Bound

![Graph showing the comparison between SPN Bound and Trivial Bound over rounds. The graph plots the degree of the function (deg(F)) against the number of rounds. The degree increases rapidly after a certain point, indicating the effectiveness of the SPN Bound.]
Application to AES

One round:

\[ MC \circ SR \circ SB \circ AK. \]

- **AK**: AddRoundKey
- **SB**: SubBytes (Sboxes of degree 7)
- **SR**: ShiftRows
- **MC**: MixColumns
The Super Sbox technique

Two rounds:

\[ R^2 = MC \circ SR \circ SB \circ AK \circ MC \circ SR \circ SB \circ AK. \]

Equivalently:

\[ R^2 = MC \circ SR \circ SB \circ AK \circ MC \circ SB \circ SR \circ AK. \]

Denote:

SuperSbox = SB \circ AK \circ MC \circ SB.

Then:

\[ R^2 = MC \circ SR \circ SuperSbox \circ SR \circ AK. \]
Bound on up to 4 rounds

**SuperSbox**: $F_{2}^{32} \rightarrow F_{2}^{32}$: Two non-linear layers composed of Sboxes of degree 7, separated by a linear layer.

$$\deg(\text{SuperSbox}) \leq 32 - \frac{32 - 7}{7} \leq 28.$$ 

(Trivial Bound: $\deg(R^2) \leq 7^2 = 49$ !!!!)

**Bound for $r$ rounds:**

$$\deg(R^r) = \deg(R^{r-1} \circ R) \leq 128 - \frac{128 - \deg(R^{r-1})}{7}.$$

- $r = 3$: $\deg(R^3) \leq 113$
- $r = 4$: $\deg(R^4) \leq 125$
Exercice (JH hash function \([Wu\ 08]\))

42 rounds of a 1024-bit permutation \(R\)

\(S\): Permutation over \(\mathbb{F}_2^4\) of degree 3.

What is the degree after 2 rounds?
Some first bounds on the degree

A bound on the degree of SPN constructions

Influence of the inverse permutation
An observation on SHA-3

\[ \chi^{-1}(x_0, \ldots, x_4) = (x_0 + x_2 + x_4 + x_1x_2 + x_1x_4 + x_3x_4 + x_1x_3x_4, \]
\[ x_0 + x_1 + x_3 + x_0x_2 + x_0x_4 + x_2x_3 + x_0x_2x_4, \]
\[ x_1 + x_2 + x_4 + x_0x_1 + x_1x_3 + x_3x_4 + x_0x_1x_3, \]
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\[ x_1 + x_3 + x_4 + x_0x_1 + x_0x_3 + x_2x_3 + x_0x_2x_3). \]

Observation of \cite{Duan-Lai11}: \[ \delta_2(\chi^{-1}) = 3. \]
An interesting property

**Question:** Is $\delta_2(\chi^{-1})$ related to $\deg(\chi)$?
An interesting property

**Question:** Is $\delta_2(\chi^{-1})$ related to $\deg(\chi)$?

**Theorem:** Let $F$ be a permutation on $F_2^n$. Then, for any integers $k$ and $\ell$,

$$\delta_\ell(F) < n - k \text{ if and only if } \delta_k(F^{-1}) < n - \ell.$$
Proof: We show that if

\[ \delta_\ell(F^{-1}) < n - k \] then \[ \delta_k(F) < n - \ell. \]

Let \( \pi(x) = \prod_{i \in K} F_i(x) \), with \( |K| = k \). The coefficient \( a \) of \( \prod_{j \not\in L} x_j \) in the ANF of \( \pi \) for \( |L| = \ell \),

\[
a = \sum_{x \in \mathbb{F}_2^n \atop x_j = 0, j \in L} \pi(x) \mod 2
\]

\[
= \#\{x \in \mathbb{F}_2^n : x_j = 0, j \in L \text{ and } F_i(x) = 1, i \in K\} \mod 2
\]

\[
= \#\{y \in \mathbb{F}_2^n : y_i = 1, i \in K \text{ and } F_j^{-1}(y) = 0, j \in L\} \mod 2
\]

\[
= \#\{y \in \mathbb{F}_2^n : y_i = 1, i \in K \text{ and } \prod_{j \in L}(1 + F_j^{-1}(y)) = 1\} \mod 2
\]

\[
= 0
\]

since, \( \deg \prod_{j \in L}(1 + F_j^{-1}(y)) < n - k \).
Application to SHA-3

**Corollary:** Let $F$ be a permutation on $\mathbb{F}_2^n$. Then, for any integer $\ell$

$$\delta_\ell(F) < n - 1 \text{ if and only if } \deg(F^{-1}) < n - \ell.$$  

**Case of SHA-3:** For $F = \chi^{-1}$ and $\ell = 2$,

$$\delta_2(\chi^{-1}) < 5 - 1 \text{ iff } \deg(\chi) < 5 - 2.$$
A new bound on the degree

[Boura – Canteaut IEEE-IT 13]

**Corollary:** Let $F$ be a permutation of $\mathbb{F}_2^n$ and let $G$ be a function from $\mathbb{F}_2^n$ into $\mathbb{F}_2^m$. Then, we have

$$\text{deg}(G \circ F) < n - \left\lfloor \frac{n - 1 - \text{deg } G}{\text{deg}(F^{-1})} \right\rfloor.$$
Consequence on the bound on SPN constructions

Recall the bound:

\[ \deg(G \circ F) \leq n - \frac{n - \deg(G)}{\gamma(S)}, \]

where

\[ \gamma(S) = \max_{1 \leq i \leq n_0 - 1} \frac{n_0 - i}{n_0 - \delta_i(S)}. \]

We can show that

\[ \gamma(S) \leq \max \left( \frac{n_0 - 1}{n_0 - \deg S}, \frac{n_0}{2} - 1, \deg S^{-1} \right). \]

For the inverse of Keccak-\(f\):

\[ \gamma(\chi^{-1}) \leq 2 \]
Bound on the degree of the inverse of Keccak-$f$
Application to $\mathcal{KN}$

Higher-order differential attack due to the low degree of the round permutation.

How to “repair” the cipher?

[Nyberg 93]:

Replace $S$ by the inverse of a quadratic permutation.

- The quadratic permutation and its inverse will have the same properties regarding differential and linear attacks.
- The quadratic permutation is not involved neither in the encryption, nor in the decryption.
The $\mathcal{KN}'$ cipher

$\tilde{\sigma} : \mathbb{F}_2^8 \rightarrow \mathbb{F}_2^8$

$x \mapsto t \circ \sigma (e(x)))$

e : $\mathbb{F}_2^8 \rightarrow \mathbb{F}_2^9$ affine expansion

t : $\mathbb{F}_2^9 \rightarrow \mathbb{F}_2^8$ truncation

$x : \sigma(x) = x^{171}$ (the inverse of $x^3$ over $\mathbb{F}_{2^9}$)

deg($\tilde{S}$) = 5

$\mathbb{F}_2^{32} \times \mathbb{F}_2^{32} \rightarrow \mathbb{F}_2^{32} \times \mathbb{F}_2^{32}$

$(x, y) \mapsto (y, x + \mathcal{L}' \circ \tilde{S} (\mathcal{L}(x) + k_i))$
Attacking $\mathcal{KN}'$

Jakobsen-Knudsen attack:

\[
\deg(y_4) \leq 5 \times 5 \times 5
\]
Attacking $\mathcal{KN}'$

Jakobsen-Knudsen attack:

$\text{deg}(y_4) \leq 5 \times 5 \times 5$

unfeasible
Attacking $\mathcal{KN}'$

Jakobsen-Knudsen attack:

$\deg(y_4) \leq 5 \times 5 \times 5$

unfeasible

Set,

$F_k(x) = \mathcal{L}' \circ \tilde{S} (\mathcal{L}(x) + k)$.

Then,

\[
\begin{align*}
y_0 &= c \\
y_1 &= x + F_{k_1}(y_0) := x + c' \\
y_2 &= F_{k_2}(x + c') + c \\
y_3 &= F_{k_3}(F_{k_2}(x + c') + c') + x + c' \\
y_4 &= y_2 + F_{k_4}(y_3)
\end{align*}
\]
Application of the new bound

\[ y_4 + y_2 = G \circ S(x) \]

Using the bound with the inverse:

\[
\deg(G \circ S) < 36 - \left\lfloor \frac{35 - \deg(G)}{2} \right\rfloor,
\]

From a previous Corollary: \((\deg(G) \leq 22)\), thus

\[
\deg(y_4) \leq \deg(G \circ S) \leq 29
\]
Application of the new bound

\[ y_4 + y_2 = G \circ S(x) \]

Using the bound with the inverse:

\[ \text{deg}(G \circ S) < 36 - \left\lfloor \frac{35 - \text{deg}(G)}{2} \right\rfloor, \]

From a previous Corollary: \((\text{deg}(G) \leq 22)\), thus

\[ \text{deg}(y_4) \leq \text{deg}(G \circ S) \leq 29 \]

**Distinguisher** on 5 rounds of \(\mathcal{KN}'\) with data complexity \(2^{30}\) that improves the generic distinguisher.
Generalization to balanced functions (not permutations)

**DES:** Eight different $6 \times 4$ Sboxes.

Can the bound be **generalized** to balanced functions from $\mathbb{F}_2^n$ to $\mathbb{F}_2^m$, with $m < n$?
Generalization to balanced functions (not permutations)

**DES:** Eight different $6 \times 4$ Sboxes.

Can the bound be generalized to balanced functions from $\mathbb{F}_2^n$ to $\mathbb{F}_2^m$, with $m < n$?

**Corollary:** Let $F$ be a balanced function from $\mathbb{F}_2^n$ into $\mathbb{F}_2^m$ and $G$ be a function from $\mathbb{F}_2^m$ into $\mathbb{F}_2^k$. For any permutation $F^*$ expanding $F$, we have

$$\deg(G \circ F) < n - \left\lfloor \frac{n - 1 - \deg G}{\deg(F^* - 1)} \right\rfloor.$$